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On the Numerical Range of a Generalized Derivation

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Abstract

We examine the relationship between the numerical range of the restriction of a generalized derivation to a norm ideal J and that of its implementing elements.

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1 Introduction

Given a Banach algebra $\mathscr{A}$, $\mathscr{A}^*$ the dual of $\mathscr{A}$, $S(\mathscr{A}) = \{x \in \mathscr{A} : \|x\| = 1\}$, the unit sphere, and $x \in S(\mathscr{A})$, let $D(x, \mathscr{A}) = \{f \in \mathscr{A}^* : f(x) = 1 = \|f\|\}$.

The Hahn-Banach theorem guarantees that $D(x, \mathscr{A})$ is non empty for each $x \in S(\mathscr{A})$. The elements of $D(I, \mathscr{A})$, I, the identity in $\mathscr{A}$, are called normalized states or simply states.
For $a \in \mathscr{A}$, and $x \in S(\mathscr{A})$, we define $V(x, a, \mathscr{A}) = \{f(ax) : f \in D(x, \mathscr{A})\}$.
The numerical range of a is the set $V(a, \mathscr{A}) = \bigcup\{V(x, a, \mathscr{A}) : x \in S(\mathscr{A})\}$.
Given a Banach space $\mathscr{H}$,we may consider the Banach algebra $\mathscr{A} = L(\mathscr{H})$ and define define the spatial numerical range of A by
$W(A; L(\mathscr{H})) = \{f(Ax) : f \in \mathscr{H}^*, x \in \mathscr{H}, and \, \|f\| = \|x\| = 1 = f(x)\}$
We first give some basic properties of the numerical range .
Bonsal [4], has shown that $V(a, \mathscr{A}) = V(I, a, \mathscr{A})$, and for each $a \in \mathscr{A}, V(a, \mathscr{A})$ is a compact convex subset of $\mathbb{C}$.

Lemma 1. *$V(x, a, \mathscr{A}) = \{f(ax) : f \in D(x, \mathscr{A})\}$ is convex.*

Proof. Let $\lambda_1, \lambda_2 \in V(x, a, \mathscr{A})$. Then there exist support functionals $f_1, f_2 \in D(x, \mathscr{A})$ such that $\lambda_1 = f_1(ax)$,$\lambda_2 = f_2(ax)$.
Define f on $D(x, \mathscr{A})$ by $f(ax) = tf_1(ax) + (1-t)f_2(ax), t \in (0, 1)$. We need to show that$f \in D(I, \mathscr{A})$ Clearly f is linear and $|f(ax)| = |tf_1(ax) + (1-t)f_2(ax)| \leq t\,|f_1(ax)| + (1-t)\,|f_2(ax)| \leq t\,\|f_1\|\,\|ax\| + (1-t)\,\|f_2\|\,\|ax\| = \|ax\| \Rightarrow \|f\| \leq 1$.
Also, $f(x) = tf_1(x) + (1-t)f_2(x) = 1$
$\Rightarrow \|f\| \geq 1$
Thus $f \in D(I, \mathscr{A})$ which is convex and hence $V(x, a, \mathscr{A})$ is convex. □

For $a \in \mathscr{A}$, we define the left multiplication operator $L_a : \mathscr{A} \to \mathscr{A}$ by $L_a(x) = ax, \forall x \in \mathscr{A}$ and $\|L_a\| = sup\{\|ax\| : x \in \mathscr{A}, \|x\| \leq 1\}$
L_a is a linear operator in $\mathscr{A}$ and also a bounded operator since
$\|L_a\| = sup\{\|ax\| : x \in \mathscr{A}, \|x\| \leq 1\} \leq \|a\|$.
$L_a(\mathscr{A})$ will denote the set of all left multiplication operators on the algebra $\mathscr{A}$ as a ranges on $\mathscr{A}$ .This set is a normed algebra.

The algebraic numerical range of $L_a \in L_a(\mathscr{A})$ is the non-empty set:
$V(L_a; L_a(\mathscr{A})) = \{f(L_a); f \in L_a(\mathscr{A})^*, f(L_e) = 1 = \|f\|\}$.
Similarly the right multiplication operator for $b \in \mathscr{A}$ is defined as ;
$R_b : \mathscr{A} \to \mathscr{A}, x \to xb$
We note that $\forall x \in \mathscr{A}$ and fixed $a, b \in \mathscr{A}$, $\Delta_{a,b}(x) = L_a(x) - R_b(x) = ax - xb$, is the generalized derivation induced by $a, b \in \mathscr{A}$.
In [3], it is shown that for any Banach algebra $\mathscr{A}$, $\|L_a\| = \|a\| = \|R_a\|$ and that $V(a; \mathscr{A}) = V(L_a; L(\mathscr{A})) = V(R_a; L(\mathscr{A}))$, $L(\mathscr{A})$ the algebra of the bounded linear operators on $\mathscr{A}$.

Lemma 2. *For $a \in \mathscr{A}, L_a \in L_a(\mathscr{A}), \|L_a\| = \|a\| = \|R_a\|$*

Proof.

$$\begin{aligned}\|L_a\| &= sup\,\{\|L_a(x)\| : \|x\| = 1\}\\ &= sup\,\{\|ax\| : \|x\| = 1\}\\ &\leq \|a\|\,\|x\|\\ &\Rightarrow \|L_a\| \leq \|a\| \,......................(i)\end{aligned}$$

If $\mathscr{A}$ has unit e, we have $L_a(e) = ae = a$ which implies $\|a\| = \|L_a(e)\| \leq \|L_a\|\,\|e\| = \|L_a\| \Rightarrow \|L_a\| \geq \|a\|$ ....(ii)
From (i) and (ii) equality follows.
Similarly we obtain $\|R_a\| = \|a\|$. □

Lemma 3. *For $a \in \mathscr{A}, V\,(a;\mathscr{A}) = V(L_a; L(\mathscr{A})) = V(R_a; L(\mathscr{A}))$*

Proof. Let $\lambda \in V(a : \mathscr{A})$, Then there exist $f \in S(\mathscr{A})$ such that $f(a) = \lambda$
Now define F on $L(\mathscr{A})$ by
$F(L_a) = f(ax)$, for all $L_a \in L(\mathscr{A})$.
Clearly F is linear since

$$\begin{aligned}F\,(\alpha L_a + \beta L_b) =& f\,(\alpha ax + \beta bx)\\ =& f(\alpha ax) + f(\beta bx)\\ =& \alpha f(ax) + \beta f(bx)\\ =& \alpha F(L_a) + \beta F(L_b), a, b \in \mathscr{A}, \alpha, \beta \in \mathbb{C}\end{aligned}$$

f is also bounded and positive since
$\|F(L_a)\| = sup\,\{\|f(ax)\|\} \leq \|f\|\,\|ax\| = c\,\|L_a\|$.
Also $F(L_e) = f(ex) = f(x) = 1$ and $\|F\| = 1$.
So F as defined is a positive linear functional on $\mathscr{A}$.
Take a finite rank operator $b \in L(\mathscr{A})$ defined by
$bx = g(x)a$, for all $x \in \mathscr{A}, g \in S(\mathscr{A})$. Clearly $\|b\| = 1$ and $F(b) = f(bx) = f(g(x)a) = g(x)f(a) = \lambda$. Hence $V\,(a;\mathscr{A}) \subseteq V(L_a; L(\mathscr{A}))$
Conversely we show that $V(L_a; L(\mathscr{A})) \subseteq V\,(a;\mathscr{A})$
Let $\lambda \in V(L_a; L(\mathscr{A}))$. Then there exists a state $f \in L(\mathscr{A})^*$ such that $f(L_a) = \lambda$
Define a functional $h \in \mathscr{A}^*$ by $h(a) = f(L_a)$. Then :

$$\begin{aligned}h\,(\alpha a + \beta b) =& f\,(\alpha L_a + \beta L_b)\\ =& f(\alpha L_a) + f(\beta L_b)\\ =& \alpha f(L_a) + \beta f(L_b)\\ =& \alpha h(a) + \beta h(b)\end{aligned}$$

$\Rightarrow h$ is linear and bounded. h is also positive since $h(a^*a) = f(L_a^* L_a) \geq 0$
Furthermore h is of norm 1 since $h(e) = f(L_e) = 1$ and
$1 = |h(e)| \leq \|h\| \|e\| \Rightarrow \|h\| \geq 1$. We also have

$$\begin{aligned}\|h\| &= sup\{|h(a)| : \|a\| = 1\} \\ &= sup\{|f(L_a)| : \|L_a| = 1\} \\ &\leq \|f\| \\ &= 1\end{aligned}$$

Thus h is a state on $\mathscr{A}^*$ and so $V(L_a; L(\mathscr{A})) \subseteq V(a; \mathscr{A})$ □

2 NORM IDEALS

Let X and Y be Banach algebras. $L(X)$ and $L(Y)$, the algebra of all bounded linear operators on X and Y respectively.
Let $(J, \|.\|_J)$ be a norm ideal on $L(Y, X)$, the algebra of all bounded linear operator from Y to X such that:

i) $(J, \|.\|_J)$ is a Banach space

ii) If $A \in L(X), T \in J, B \in L(Y)$ then $ATB \in J$,and $\|ATB\|_J \leq \|A\| \|T\|_J \|B\|$

iii) $\|T\| \leq \|T\|_J$, $T \in J$and

iv) $\|T\|_J = \|T\|$, for T a rank- one operator.

If $A \in L(X), B \in L(Y)$ and $T \in J$, then the operators L_A, R_B and $L_A - R_B$ are all bounded linear operators on $L(J)$, the space of all bounded linear operators from J to J, where:
$L_A T = AT$, the left multiplication operator,
$R_B T = TB$, the right multiplication operator and
$(L_A - R_B)T = AT - TB$, the generalized derivation.The following lemma will hold.

Lemma 4. $V(A : L(X)) = V(L_A : L(J))$

Proof. Let $\lambda \in V(A : L(X))$. Then there exist $f \in L(X)^*$ such that
$\lambda = f(A), and, f(I_{L(X)}) = 1 = \|f\|$
Let $\mathscr{A}_0 = \{L_A : A \in L(X), L_A(T) = AT, T \in J\} \subseteq L(J)$.
$\mathscr{A}_0$ is a linear subspace of $L(X)$.
On $\mathscr{A}_0^*$, define a linear functional g such that $g(L_A) = f(A)$. Clearly g as defined is a state and the Hahn-Banach theorem guarantees the existence of its extension on $L(J)$. Hence, $V(A : L(X)) \subseteq V(L_A : L(J))$
$\Leftarrow$: suppose $\lambda \in V(L_A : L(J)$. Then $\exists\ f \in L(J)^*$ such that $f(L_A) = \lambda$ and

$f(I_{L(J)}) = 1 = \|f\|$.
Define a linear operator h on $L(X)^*$ by $h(A) = f(L_A)$. Then $h(I) = f(I_{L(J)}) = 1$.
h is thus a state on $L(X)^*$ and $V(L_A : L(J)) \subseteq V(A : L(X))$ □

3 Norm of L_A and R_B in $(J, \|.\|_J)$

Lemma 5. $\|L_A\|_J = \|A\|$

Proof. Condition (ii) above on the definition of a norm ideal implies that L_A and R_B are bounded linear operators on $(J. \|.\|_J)$ and

$$\begin{aligned}\|L_A\|_J &= Sup\{\|AX\| : \|X\|_J = 1, X \in J\}\\ &\leq \|A\| \, \|X\|_J\\ &= \|A\|\end{aligned}$$

Condition (iii) implies $\|L_A\|_J \geq \|A\|$.
It therefore follows that $\|L_A\|_J = \|A\|$
Similarly $\|R_B\|_J = \|B\|$ □

4 Numerical range of the generalized derivation in the norm ideal J

In the past, generalized derivations , their properties and their restrictions to norm ideals have been investigated by many authors. For example,their spectra have been characterized in [7] and [8]. The famous results on the norms of inner derivation and the generalized derivation as obtained by Stampfli [6] using maximal numerical range have ever since provided a crucial lead in defining of norms of elementary operators.We recall the works of Kyle [9] who examines the relationship between the numerical range of an inner derivation, and that of its implementing element.
In his paper, Magajna [2] gives the essential numerical range of the the generalized derivation defined on the Hilbert-Schmidt class in terms of the numerical and the essential numerical ranges of the implementing operators. Shaw [10] in particular, established that the algebra numerical range of a generalized derivation restricted to a norm ideal J is equal to the difference of the algebra numerical ranges of the implementing operators provided that J contains all finite rank operators and is suitably normed . With slight modification we obtain an alternative proof to Shaw's result.

Lemma 6. *Let J be as defined above. Then for $A \in L(X), B \in L(Y), V(\Delta_{A,B} : L(J)) = V(A : L(X)) - V(B : L(Y))$*

Proof. Let $\lambda \in V\left(\Delta_{A,B} : L(J)\right)$. This implies there exist $f \in L(J)^*$ such that $f(\Delta_{A,B}) = \lambda$ and $f\left(I_{L(J)}\right) = 1 = \|f\|$
Let $\mathscr{A}_0 = \{L_A : A \in L(X), L_A(T) = AT, T \in J\} \subseteq L(J)$ and
$\mathscr{A}_1 = \{R_B : B \in L(Y), R_B(T) = TB, T \in J\} \subseteq L(J)$ i.e. the set of the left and right multiplication operators respectively in $L(J)$. These are linear subspaces of $L(X)$ and $L(Y)$ respectively.Let also $S(L(J)) = \left\{f \in L(J)^* : f\left(I_{L(J)}\right) = 1 = \|f\|\right\}$

$$\begin{aligned}\lambda = f\left(\Delta_{A,B} : L(J)\right) &= \{f\left(L_A - R_B : f \in S(L(J))\right)\} \\ &= \left\{f(L_A) : f \in L(X)^*, f(I_{L(X)}) = 1 = \|f\|\right\} \\ &- \left\{f(R_B) : f \in L(Y)^*, f(I_{L(Y)}) = 1 = \|f\|\right\} \\ &= V\left(L_A : L_A \in L\left(J\right)\right) - V\left(R_B : R_B \in L\left(J\right)\right) \\ &\in V\left(A : L(X)\right) - V\left(B : L(Y)\right)\end{aligned}$$

To prove the reverse inclusion, we make use of the spatial numerical range.Choose λ in $W\left(A : L(X)\right)$ and μ in $W\left(B : L(Y)\right)$. Then we can find functionals f and g in $L(X)^*, L(Y)^*$ such that
$\|f\| = \|x\| = f\left(x\right) = 1$, with $f(Ax) = \lambda$ and
$\|g\| = \|y\| = g(y) = 1$, with $g(By) = \mu$
Let X be a rank one operator in J such that $Xz = g(z)x, \forall z \in Y$,
Also define F in $L(J)^*$ by $F(T) = f(Ty), \forall T \in L(J)$
Then $F(X) = f(Xy) = fg(y)x = g(y)f(x) = 1$,
$F(I) = f(Iy) = fg(y)x = g(y)f(x) = 1$ and
$|F(T)| \leq \|f\| \, \|T\|_J \, \|Y\| = \|T\|_J$
Clearly $\|F\|_J = \|X\|_J = 1$ and $\left(I_{L(J)}, F\right) \in L(J) \times L(J)^*$
Thus,

$$\begin{aligned}F(\Delta_{A,B}(X)) &= F\left(AX - XB\right) \\ &= f\left(AX - XB\right)y \\ &= f(AXy) - f(XBy) \\ &= f(g(y)Ax) - f(g(By)x) \\ &= f(Ax)g(y) - f(x)g(By) \\ &= \lambda - \mu \\ &\in \{W\left(A : L(X)\right) - W\left(B : L(Y)\right)\}\end{aligned}$$

Now

$$\begin{aligned}V\left(\Delta_{A,B}; L(J)\right) &= \overline{co}W\left(\Delta_{A,B}; L(J)\right) \\ &\supseteq \overline{co}\left\{W\left(A; L(X)\right) - W\left(B; L(Y)\right)\right\} \\ &= \overline{co}\left\{W\left(A; L(X)\right)\right\} - \overline{co}\left\{W\left(B; L(Y)\right)\right\} \\ &= V\left(A; L(X)\right) - V\left(B; L(Y)\right)\end{aligned}$$

Thus $\{V\,(A; L(X)) - V(B; L(Y))\} \subseteq V(\Delta_{A,B}; L(J))$ □

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